Malaysian Journal of Mathematical Sciences 10(3): 383–398 (2016)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

The Repeated Procedure PRMZSS1 for Estimating the Polynomial Zeros Simultaneously

Mansor Monsi[1], Nasruddin Hassan*[2], and Syaida Fadhilah Mohammad Rusli[1]

[1]*Mathematics Department, Faculty of Science, Universiti Putra Malaysia, Malaysia*

[2]*School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Malaysia*

E-mail: nas@ukm.edu.my

**Corresponding author*

ABSTRACT

Our previous method that is PMZSS1 has a rate of convergence of at least eight. The aim of repeating the steps in PMZSS1 is to yield a better rate of convergence.The resulting method is called the repeated midpoint zoro PRMZSS1 where its rate of convergence is at least $7r+1$ with $r \geq 1$.The proof of this result is detailed in the convergence analysis of PRMZSS1. Numerical results and comparison with the existing procedures of PZSS1 and PMZSS1 are included to confirm our theoretical results, where the rate of convergence of PZSS1 and PMZSS1 are four and eight respectively.

Keywords: Convergence rate, estimating the zeros, simple zeros, simultaneous approximation.

1. Introduction

The establishment of the procedure PRMZSS1 is from the ideas of the procedures IRSS1 and IMZSS1 in Monsi and Wolfe (1988), Rusli et al. (2011) and Jamaludin et al. (2014b) respectively. The rate of convergence of IRSS1 is at least $(2r + 1)$ $(r \geq 1)$,while the rate of convergence of IMZSS1 is at least eight (Rusli (2011)). In our proposed procedure PRMZSS1, the steps 2-4 in IMZSS1 (Rusli (2011)) are repeated r times. As a result, the convergence analysis in section 3 shows that the convergence rate of PRMZSS1 is at least $(7r+1)$ $(r \geq 1)$. The rapid rate of convergence obtained is very impressive. For a special case, that is for $r = 1$, the procedure PRMZSS1 is reduced to PMZSS1 which has a rate of convergence of at least eight (see algorithm PRMZSS1 with $l = 1$, algorithm PMZSS1 and Theorem 3.1 below).

Other ideas for upgrading the performances of the convergence of polynomials are by Bakar et al. (2012), Jamaludin et al. (2013a, 2014a,b, 2013b,c,d), Monsi et al. (2014), Sham et al. (2013a,b,c) and Rusli et al. (2014a,b).Recent discussions on the rate of convergence can be found in Hanapiah et al. (2015a,b), Monsi and Hassan (2015), Monsi et al. (2015a,b), Rusli et al. (2015a,b,c), Sham et al. (2015), and Jamaludin et al. (2015a,b,c).

In this paper we will propose a new method called the repeated midpoint zoro PRMZSS1 by repeating a few steps of the midpoint zoro PMZSS1 procedure. Section 2 will cover the current literature survey on the estimation of polynomial zeros, while Section 3 will be on the convergence of the proposed new procedure. Chapter 4 will illustrate the comparison of results and conclusion, which will be followed by acknowledgment in Section 5.

2. Estimating The Zeros Simultaneously

Let $p : C \to C$ be a polynomial function for the set of real numbers C, given by

$$p(x) = \sum_{i=0}^{n} a_i x^i = \prod_{i=1}^{n} (x - x_i^*) \quad (n > 1) \tag{1}$$

where $a_n = 1$. This section contains several point iterative procedures for estimating a set of simple zeros $x^* \in (x_1^*, ..., x_n^*)^T$ of p simultaneously. Suppose that from (1)

$$p(x) = \prod_{j=1}^{n} (x - x_j^*) = 0. \tag{2}$$

Then, we define

$$p_i(x) = \prod_{j=1}^{i-1} (x - x_j^*) \prod_{i+1}^{n} (x - x_j^*). \tag{3}$$

By (2), if, for $i = 1, ..., n,\ x_i \neq x_j (j \neq i)$, then

$$x_i^* = x_i - \frac{p(x_i)}{\prod_{j \neq i} (x_i - x_j^*)}. \tag{4}$$

Suppose that, for x_j is an approximation of x_j^* or $x_j \approx x_j^*$, thus by (4)

$$x_i^* \approx x_i - \frac{p(x_i)}{\prod_{j \neq i} (x_i - x_j)}\ (i = 1, ..., n). \tag{5}$$

This gives rise to the procedure defined by

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j \neq i} (x_i^{(k)} - x_j^{(k)})} \quad (i = 1, ..., n)\ (k \geq 0). \tag{6}$$

Procedure (6) is then modified by Alefeld and Herzberger (1983) and defined as follows:

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=i}^{i-1} (x_i^{(k)} - x_j^{(k+1)}) \prod_{j=i+1}^{n} (x_i^{(k)} - x_j^{(k)})} \quad (i = 1, ..., n)\ (k \geq 0), \tag{7}$$

We will use equation (7) as a basic reference of our new procedure in section 3. The analysis of rate of convergence in Ortega and Rheinboldt (1970) is used in order to measure of the asymptotic convergence rate of the procedure. The approaches of analysing the convergence can also be found in Alefeld (1977), Petkovic (1982), Petkovic and Milosevic (2005), and Alefeld and Herzberger (1983). The following is the algorithm of PMZSS1 where its rate of convergence is at least eight (Rusli (2011)).

Algorithm PMZSS1

Step 1: Set $k \geq 0$

Step 2: $x_i^{(k,0)} = x_i^{(k)}$ $(i = 1, ..., n)$,

Step 3: $x_i^{(k,1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=i}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,0)})}$, $(i = 1, ..., n)$,

Step 4: $x_i^{(k,2)} = x_i^{(k,1)} - \frac{p(x_i^{(k,1)})}{\prod_{j=i}^{i-1}(x_i^{(k,1)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k,1)} - x_j^{(k,2)})}$, $(i = n, ..., 1)$,

Step 5: $x_i^{(k,3)} = x_i^{(k,2)} - \frac{p(x_i^{(k,2)})}{\prod_{j=i}^{i-1}(x_i^{(k,2)} - x_j^{(k,3)}) \prod_{j=i+1}^{n}(x_i^{(k,2)} - x_j^{(k,2)})}$, $(i = 1, ..., n)$,

Step 6; $x_i^{(k+1)} = x_i^{(k,3)}$ $(i = n, ..., 1)$.

Step 7: $\left| x_i^{(k+1)} - x_i^* \right| < \varepsilon$ $(i = 1, ..., n)$ then stop, else go to Step 2 with $k = k + 1$, where ϵ is a small number such as 10^{-12}.

The advantages of the above algorithm are:

1. the points $x_i^{(k,1)}$ in Step 3 are being used in Step 4 with new function evaluations $p(x_i^{(k,1)})$ backwardly,
2. the new points $x_i^{(k,2)}$ generated from Step 4 are being used in Step 5 with new function evaluations $p(x_i^{(k,2)})$, and

$$x_i^{(k,1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=i}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,0)})} \quad (i = 1, ..., n) \quad \text{(8a)}$$

$$x_i^{(k,2)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=i}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,2)})} \quad (i = n, ..., 1) \quad \text{(8b)}$$

$$x_i^{(k+1)} = x_i^{(k+2)} \quad (i = 1, ..., n) \quad \text{(8c)}$$

Further discussions on the rate of convergence can be found in Petkovic and Milosevic (2005) and Petkovic (1982).

3. Higher Order of Convergences Procedures

A new procedure which is called the repeated midpoint zoro symmetric single-step procedure PRMZSS1 where Steps 2-4 in PMZSS1 are repeated r times, is defined by:

Algorithm PRMZSS1 Step 1: For $l = 1, ..., r,\ k \geq 0,$

$$x_i^{(k,3l-3)} = x_i^{(k)}\ (i = 1, ..., n) \tag{9a}$$

$$x_i^{(k,3l-2)} = x_i^{(k,3l-3)} - \frac{p(x_i^{(k,3l-3)})}{\prod_{j=i}^{i-1}(x_i^{(k,3l-3)} - x_j^{(k,3l-2)}) \prod_{j=i+1}^{n}(x_i^{(k,3l-3)} - x_j^{(k,3l-3)})} \quad (i = 1, ..., n) \tag{9b}$$

$$x_i^{(k,3l-1)} = x_i^{(k,3l-2)} - \frac{p(x_i^{(k,3l-2)})}{\prod_{j=i}^{i-1}(x_i^{(k,3l-2)} - x_j^{(k,3l-2)}) \prod_{j=i+1}^{n}(x_i^{(k,3l-2)} - x_j^{(k,3l-1)})} \quad (i = n, ..., 1) \tag{9c}$$

$$x_i^{(k,3l)} = x_i^{(k,3l-1)} - \frac{p(x_i^{(k,3l-1)})}{\prod_{j=i}^{i-1}(x_i^{(k,3l-1)} - x_j^{(k,3l)}) \prod_{j=i+1}^{n}(x_i^{(k,3l-1)} - x_j^{(k,3l-1)})} \quad (i = 1, ..., n) \tag{9d}$$

Step 2:

$$x_i^{(k+1)} = x_i^{(k,3r)} \quad (i = 1, ..., n) \tag{9e}$$

$$k = k + 1 \tag{9f}$$

Step 3: If$|x_i^{(k+1)} - x_i^*| \leq \varepsilon\ (i = 1, ..., n),$ then stop, else go to step 2

This procedure has the following attractive features:

- The terms $\prod_{j=i}^{i-1}(x_i^{(k,3l-3)} - x_j^{(k,3l-2)})\ (i = 2, ..., n)\ (l \geq 1)(k \geq 0)$ which are computed in (9b) can be reused in (9c).
- The terms $\prod_{i+1}^{n}(x_i^{(k,3l-2)} - x_j^{(k,3l-1)})\ (i = n-1, ..., 1)\ (l \geq 1)(k \geq 0)$ which are computed in (9c)can be reused in (9d).

The following theorem will show the rate of convergence of our proposed method.

Theorem 3.1. If (i) $p : C \to C$ defined by (1) with distinct zeros x_i^* $(i = 1, ..., n)$; (ii) $\left|x_i^{(0)} - x_i^*\right| \leq \frac{\theta d}{2n-1}$ $(i = 1, ..., n)$ with $0 < \theta < 1$ and $d = min\left|x_i^* - x_j^*\right|$ $j \neq i$, the sequences $x_i^{(k)}$ are from (9), then $x_i^{(k)} \to x_i^*(k \to \infty)$ and $O_R(PRMZSS1, x^*) \geq 7r + 1$ $(r \geq 1)$.

Proof. Based on $q_{(1,i)}, q_{(2,i)}$ and $q_{(3,i)}$ in Monsi et al. (2012), we may define the following functions as follows. For $l = 1, ..., r$, let

$$q_{3l-2,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,3l-2)}) \prod_{m=i+1}^{n}(x - x_m^{(k,3l-3)}), \tag{10}$$

$$q_{3l-1,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,3l-2)}) \prod_{m=i+1}^{n}(x - x_m^{(k,3l-1)}), \tag{11}$$

and

$$q_{3l,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,3l)}) \prod_{m=i+1}^{n}(x - x_m^{(k,3l-1)}). \tag{12}$$

Then, by considering functions $\phi_{(1,i)}, \phi_{(2,i)}$ and $\phi_{(3,i)}$ in Monsi et al.(2012), let for $i = 1, ..., n$; $l = 1, ..., r$,

$$\begin{aligned}\phi_{3l-2,i}(x) = q_{3l-2,i}(x) &+ \sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,3l-3)})q_{3l-2,i}(x)}{q'_{3l-2,i}(x_j^{(k,3l-2)})(x - x_j^{(k,3l-2)})} \\ &+ \sum_{i+1}^{n} \frac{p_i(x_j^{(k,3l-3)})q_{3l-2,i}(x)}{q'_{3l-2,i}(x_j^{(k,3l-3)})(x - x_j^{(k,3l-3)})'}\end{aligned} \tag{13}$$

$$\begin{aligned}\phi_{3l-1,i}(x) = q_{3l-1,i}(x) &+ \sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,3l-2)})q_{3l-1,i}(x)}{q'_{3l-1,i}(x_j^{(k,3l-2)})(x - x_j^{(k,3l-2)})} \\ &+ \sum_{i+1}^{n} \frac{p_i(x_j^{(k,3l-1)})q_{3l-1,i}(x)}{q'_{3l-1,i}(x_j^{(k,3l-1)})(x - x_j^{(k,3l-1)})'}\end{aligned} \tag{14}$$

and

$$\phi_{3l,i}(x) = q_{3l,i}(x) + \sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,3l)})q_{3l,i}(x)}{q'_{3l,i}(x_j^{(k,3l)})(x - x_j^{(k,3l)})} + \sum_{i+1}^{n} \frac{p_i(x_j^{(k,3l)})q_{3l,i}(x)}{q'_{3l,i}(x_j^{(k,3l-1)})(x - x_j^{(k,3l-1)})'} \tag{15}$$

where $p_i(x)$ is defined by (3).By Lemma 1 and Lemma 2 of Monsi et al.(2012),with $q_i = q_{3l-2,i}, \check{x}_i = x_i^{(k,3l-3)}, \hat{x}_i = x_i^{(k,3l-3)}, \bar{x}_i = x_i^{(k,3l-2)}, \phi_i = \phi_{3l-2,i}$ $(i = 1, ..., n)$,then for $k \geq 0$,

$$w_i^{(k,3l-2)} = w_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} a_{ij}^{(k,3l-2)} w_j^{(k,3l-2)} + \sum_{i+1}^{n} a_{ij}^{(k,3l-3)} w_j^{(k,3l-3)} \right\}, \tag{16}$$

where

$$w_i^{(k,s)} = x_i^{(k,s)} - x_i^* \ (s = 0, 1, ..., 3r)$$

$$a_{i,j}^{(k,3l-2)} = \frac{\prod_{m \neq i,j}(x_j^{(k,3l-2)} - x_m^*)}{q'_{3l-2,i}(x_j^{(k,3l-2)})(x_j^{(k,3l-2)} - x_i^{(k,3l-3)})'} \quad (j = 1, ..., i-1), \tag{17}$$

and

$$a_{i,j}^{(k,3l-3)} = \frac{\prod_{m \neq i,j}(x_j^{(k,3l-3)} - x_m^*)}{q'_{3l-2,i}(x_j^{(k,3l-3)})(x_j^{(k,3l-3)} - x_i^{(k,3l-3)})'} \quad (j = i+1, ..., n). \tag{18}$$

Using Lemma 1 and Lemma 2 of Monsi et al. (2012), with $q_i = q_{3l-1,i}, \check{x}_i = x_i^{(k,3l-2)}, \hat{x}_i = x_i^{(k,3l-1)}, \bar{x}_i = x_i^{(k,3l-2)}, \phi_i = \phi_{3l-1,i}$, we have

$$w_i^{(k,3l-1)} = w_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} \beta_{ij}^{(k,3l-2)} w_j^{(k,3l-2)} + \sum_{i+1}^{n} \beta_{ij}^{(k,3l-1)} w_j^{(k,3l-1)} \right\}, \tag{19}$$

where

$$\beta_{i,j}^{(k,3l-2)} = \frac{\prod_{m \neq i,j}(x_j^{(k,3l-2)} - x_m^*)}{q'_{3l-1,i}(x_j^{(k,3l-2)})(x_j^{(k,3l-2)} - x_i^{(k,3l-2)})} \quad (j = 1, ..., i-1; i = 1, ..., n), \tag{20}$$

and

$$\beta_{i,j}^{(k,3l-1)} = \frac{\prod_{m\neq i,j}(x_j^{(k,3l-1)} - x_m^*)}{q'_{3l-1,i}(x_j^{(k,3l-1)})(x_j^{(k,3l-1)} - x_i^{(k,3l-2)})} \;\; (j = 1,...,i-1; i = 1,...,n), \tag{21}$$

By Lemma 1 and Lemma 2 with $q_i = q_{3l,i}, \check{x}_i = x_i^{(k,3l-1)}, \hat{x}_i = x_i^{(k,3l-1)}, \bar{x}_i = x_i^{(k,3l)}, \varphi_i = \varphi_{3l,i}$, then for $k \geq 0$,

$$w_i^{(k,3l)} = w_i^{(k,3l-1)} \left\{ \sum_{j=1}^{i-1} \gamma_{ij}^{(k,3l)} w_j^{(k,3l)} + \sum_{i+1}^{n} \gamma_{ij}^{(k,3l-1)} w_j^{(k,3l-1)} \right\}, \tag{22}$$

where

$$\gamma_{i,j}^{(k,3l)} = \frac{\prod_{m\neq i,j}(x_j^{(k,3l)} - x_m^*)}{q'_{3l,i}(x_j^{(k,3l)})(x_j^{(k,3l)} - x_i^{(k,3l-1)})} \;\; (j = 1,...,i-1; i = 1,...,n), \tag{23}$$

and

$$\gamma_{i,j}^{(k,3l-1)} = \frac{\prod_{m\neq i,j}(x_j^{(k,3l-1)} - x_m^*)}{q'_{3l,i}(x_j^{(k,3l-1)})(x_j^{(k,3l-1)} - x_i^{(k,3l-1)})} \;\; (j = i+1,...,n; i = 1,...,n). \tag{24}$$

For $l = 1$, from (16)-(18) and Lemma 3 of Monsi et al.(2012) that $|w_i^{(0,1)}| \leq \theta|w_i^{(0,0)}|$ $(i = 1,...,n)$, from (19)-(21) and Lemma 3 that $|w_i^{(0,2)}| \leq \theta^3|w_i^{(0,0)}|$ $(i = 1,...,n)$,and from (22)-(24)and Lemma 3 that $|w_i^{(0,3)}| \leq \theta^7|w_i^{(0,0)}|$ $(i = 1,...,n)$.

For $l = 2$, again by using the same approaches on (16)-(18), (19)-(21), (22)-(23) and Lemma 3 of Monsi et al. (2012), the following results are obtained. For $i = 1,...,n$,

$$|w_i^{(0,4)}| \leq \theta^8|w_i^{(0,0)}|; \; |w_i^{(0,5)}| \leq \theta^{10}|w_i^{(0,0)}|; \; |w_i^{(0,6)}| \leq \theta^{14}|w_i^{(0,0)}|.$$

Then, by inductive argument on l that

$$|w_i^{(0,3l-2)}| \leq \theta^{7l-6}|w_i^{(0,0)}|; \; |w_i^{(0,3l-1)}| \leq \theta^{7l-4}|w_i^{(0,0)}|; \; |w_i^{(0,3l)}| \leq \theta^{7l}|w_i^{(0,0)}|.$$

Then, by (9e) that is $l = r$,$|w_i^{(1,0)}| = |w_i^{(0,3r)}| \leq \theta^{7r}|w_i^{(0,0)}|$.

Next, by inductive argument on k that

$$|w_i^{(k,0)}| \leq \theta^{(7r+1)^k-1}|w_i^{(0,0)}| \; (i = 1,...,n),$$

where $x_i^{(k)} \to x_i^*$ $(i = 1, ..., n)$ as $k \to \infty$. Let

$$h_i^{(k,s)} = \frac{(2n-1)}{d}|w_i^{(k,s)}| \; (i = 1, ..., n) \; (s = 0, 1, ..., 3r). \tag{25}$$

Then, by (16)-(25),

$$h_i^{(k,3l-2)} \leq \frac{1}{(n-1)} h_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,3l-2)} + \sum_{i+1}^{n} h_j^{(k,3l-3)} \right\} \; (i = 1, ..., n), \tag{26}$$

$$h_i^{(k,3l-1)} \leq \frac{1}{(n-1)} h_i^{(k,3l-2)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,3l-2)} + \sum_{i+1}^{n} h_j^{(k,3l-1)} \right\} \; (i = n, ..., 1), \tag{27}$$

and

$$h_i^{(k,3l)} \leq \frac{1}{(n-1)} h_i^{(k,3l-1)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,3l)} + \sum_{i+1}^{n} h_j^{(k,3l-1)} \right\} \; (i = 1, ..., n). \tag{28}$$

Let

$$u_i^{(1,3l-2)} = \begin{cases} 7l-5 & (i = 1, ..., n-1) \\ 7l-4 & (i = n) \end{cases}, \tag{29}$$

$$u_i^{(1,3l-1)} = \begin{cases} 7l-1 & (i = 1) \\ 7l-3 & (i = 2, ..., n-1) \\ 7l-2 & (i = n) \end{cases}, \tag{30}$$

and

$$u_i^{(1,3l)} = \begin{cases} 7l+3 & (i = 1) \\ 7l+1 & (i = 2, ..., n-2) \\ 7l+2 & (i = n-1) \\ 7l+6 & (i = n) \end{cases}, \tag{31}$$

Let for $m = 1, ..., 3r$,

$$u_i^{(k+1,m)} = \begin{cases} (7r+1)u_i^{(k,m)} + 2 & (i = 1) \\ (7r+1)u_i^{(k,m)} & (i = 2, ..., n-2) \\ (7r+1)u_i^{(k,m)} + 1 & (i = n-1) \\ (7r+1)u_i^{(k,m)} + 5 & (i = n) \end{cases}. \tag{32}$$

Then, by (29)-(32)for $k \geq 1$,

$$u_i^{(k,3l-2)} = \begin{cases} \frac{(49lr-35r+2)}{7r}(7r+1)^{k-1} - \frac{2}{7r} & (i = 1) \\ (7l-5)(7r+1)^{k-1} & (i = 2, ..., n-2) \\ \frac{(49lr-35r-2)}{7r}(7r+1)^{k-1} - \frac{2}{7r} & (i = n-1) \\ \frac{(49lr-28r+1)}{7r}(7r+1)^{k-1} - \frac{5}{7r} & (i = n) \end{cases}, \tag{33}$$

$$u_i^{(k,3l-1)} = \begin{cases} \frac{(49lr-7r+2)}{7r}(7r+1)^{k-1} - \frac{2}{7r} & (i=1) \\ (7l-3)(7r+1)^{k-1} & (i=2,...,n-2) \\ \frac{(49lr-21r+1)}{7r}(7r+1)^{k-1} - \frac{1}{7r} & (i=n-1) \\ \frac{(49lr-14r+5)}{7r}(7r+1)^{k-1} - \frac{5}{7r} & (i=n) \end{cases}, \quad (34)$$

and

$$u_i^{(k,3l)} = \begin{cases} \frac{(49lr+21r+2)}{7r}(7r+1)^{k-1} - \frac{2}{7r} & (i=1) \\ (7l+1)(7r+1)^{k-1} & (i=2,...,n-2) \\ \frac{(49lr+14r+1)}{7r}(7r+1)^{k-1} - \frac{1}{7r} & (i=n-1) \\ \frac{(49lr+42r+5)}{7r}(7r+1)^{k-1} - \frac{5}{7r} & (i=n) \end{cases}, \forall k \geq 0, \quad (35)$$

Suppose that for $i = 1, ..., n$,

$$h_i^{(0,0)} \leq h < 1.$$

By induction on i, k and l, from (25)-(35) we have the following results. For $i = 1, ..., n;\ k \geq 0$,

$$h_i^{(k,3l-2)} \leq h^{u_i^{(k+1,3l-2)}},\ h_i^{(k,3l-1)} \leq h^{u_i^{(k+1,3l-1)}},\ h_i^{(k,3l)} \leq h^{u_i^{(k+1,3l)}}.$$

where by (35)and (9e),

$$h_i^{(k+1)} = h_i^{(k+1,0)} = h_i^{(k,3r)} \leq h^{u_i^{(k+1,3r)}} (\forall k \geq 0)$$

$$h_i^{(k)} \leq h_i^{(7r+1)^k}\ (i = 1, ..., n)\ (k \geq 0). \quad (36)$$

By (25) for $s = 3r$,

$$|w_i^{(k,3r)}| = \frac{d}{(2n-1)} h_i^{(k,3r)}\ (i = 1, ..., n).$$

Then, by (9e)

$$|w_i^{(k+1)}| = |w_i^{(k+1,0)}| = \frac{d}{(2n-1)} h_i^{(k+1,0)} = \frac{d}{(2n-1)} h_i^{(k+1)}\ (i = 1, ..., n).$$

So, for $(i = 1, ..., n)$ $(k \geq 0)$,

$$|w_i^{(k)}| = \frac{d}{(2n-1)} h_i^{(k)}. \quad (37)$$

Let

$$w^{(k)} = \max_{1 \leq i \leq n} \{|w_i^{(k)}|\}, \quad (38)$$

and

$$h^{(k)} = \max_{1 \le i \le n} \{h_i^{(k)}\}. \tag{39}$$

Then, by (26)-(39)

$$w^{(k)} \le \frac{d}{(2n-1)} h^{(7r+1)^k} \ (\forall k \ge 0).$$

So,

$$\begin{aligned} R_{7r+1}(w^{(k)}) &= \lim_{k\to\infty} sup\{(w^{(k)})^{1/(7r+1)^k}\} \\ &\le \lim_{k\to\infty} sup\{\frac{d}{(2n-1)}^{1/(7r+1)^k} h\} \\ & h < 1. \end{aligned}$$

Therefore, by Ortega and Rheinboldt (1970), we thereby have

$$O_R(PRMZSS1, x_i^*) \ge (7r+1) \ (r \ge 1).\blacksquare$$

4. Numerical Result and Conclusion

The above analysis clearly showed the superiority of the PRMZSS1 procedure with a very high rate of convergence compared to any other procedures of the same class of research area. The idea of using the midpoints in the procedure PMZSS1 came from the work of Rusli et al.(2011). This midpoint factor contributes to hasten the rate of convergence of PRMZSS1 to at least $(7r + 1)$ $(r \ge 1)$.Table 1 shows the CPU times to run the three algorithms PSS1, PMZSS1 and PRMZSS1 using five test polynomials in Rusli et al. (2011), with initial points $x_i^{(0)}$ being the midpoints of the respective intervals $X_i^{(0)}$ in Monsi and Wolfe (1988). The convergence criterion used is $|x_i^{(k)} - x_i^*| < 10^{-12}$ $(i = 1, ..., n)$.

Table 1: CPU Times and Number of Iterations k and r

Polynomial	n	**PSS1**(k)	**PMZSS1**(k)	**PRMZSS1**(k,r)
1	4	0.22152(3)	0.21216(2)	0.14976(2,3)
2	6	0.32136(3)	0.29640(2)	0.23712(1,2)
3	9	0.54600(3)	0.51480(2)	0.48048(1,4)
4	5	0.27144(3)	0.26208(2)	0.19968(1,2)
5	6	0.31200(3)	0.28080(2)	0.24024(1,2)

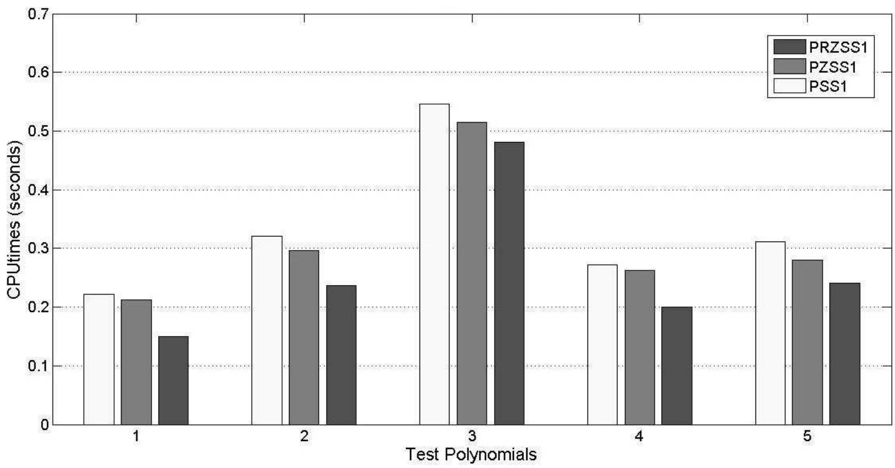

Figure 1: Comparison of CPU times

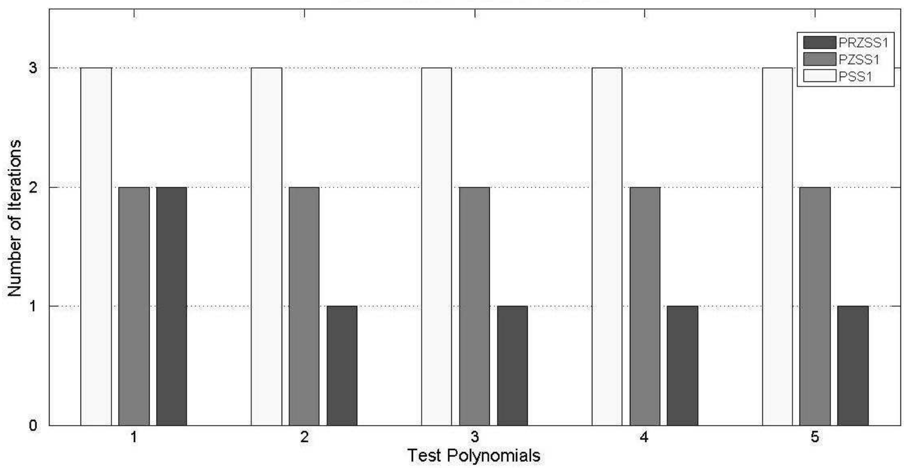

Figure 2: Comparison of Number of Iterations

The results show that the procedure PRMZSS1 is more efficient compared to PSS1 (Monsi (2012)) and PMZSS1(Rusli et al. (2011)). This can be seen clearly in Figures 1 and 2. The CPU times and number of iterations for PRMZSS1 reduced very sharply compared to both PSS1 and PMZSS1. Note that the procedure PRMZSS1 has also its own interval version that is the procedure IRMZSS1 which has been discussed in Rusli et al. (2011), where their rate of convergences are the same.

Acknowledgement

We are indebted to the anonymous reviewers whose suggestions have improved the content of this article and Universiti Kebangsaan Malaysia for the grant BKBP-FST-K005560-2014.

References

Alefeld, G. (1977). The symmetric single-step method for systems of simultaneous linear equations with intervals as coefficients. *Computing*, 18(4):329–340.

Alefeld, G. and Herzberger, J. (1983). *Introduction to Interval Computations.* Academic Press, New York.

Bakar, N. A., Monsi, M., and Hassan, N. (2012). An improved parameter regula falsi method for enclosing a zero of a function. *Applied Mathematical Sciences*, 6(28):1347–1361.

Hanapiah, N. L. M., Monsi, M., Hassan, N., and Ali, F. M. (2015a). Interval total single step procedure for bounding polynomial zeros. *Far East Journal of Mathematical Sciences*, 98(5):539–546.

Hanapiah, N. L. M., Monsi, M., Hassan, N., and Masseran, N. (2015b). On the convergence of interval total single step procedure for polynomial zeros. *Far East Journal of Mathematical Sciences*, 98(5):641–648.

Jamaludin, N., Monsi, M., and Hassan, N. (2013a). On the interval zoro symmetric single step procedure IZSS2-5D for simultaneous bounding of simple polynomial zeros. *International Journal of Mathematical Analysis*, 7(59):2941–2945.

Jamaludin, N., Monsi, M., and Hassan, N. (2014a). Interval symmetric single-step procedure IZSS2-5D for polynomial zeros. *Sains Malaysiana*, 43(7):1101–1104.

Jamaludin, N., Monsi, M., and Hassan, N. (2014b). On convergence of the interval zoro symmetric single step procedure for polynomial zeros. *International Journal of Mathematical Analysis*, 8(56):2769–2774.

Jamaludin, N., Monsi, M., Hassan, N., and Kartini, S. (2013b). On modified interval symmetric single step procedure ISS2-5D for the simultaneous inclusion of polynomial zeros. *International Journal of Mathematical Analysis*, 7(20):983–988.

Jamaludin, N., Monsi, M., Hassan, N., and Suleiman, M. (2013c). Modification on interval symmetric single-step procedure ISS-5δ for bounding polynomial zeros simultaneously. *AIP Conf. Proc.*, 1522:750–756.

Jamaludin, N., Monsi, M., Said Husain, S. K., and Hassan, N. (2013d). The interval zoro symmetric single-step procedure IZSS2-5D for the simultaneous bounding of simple polynomial zeros. *AIP Conf. Proc.*, 1557:268–271.

Jamaludin, N. A. A., Monsi, M., and Hassan, N. (2015a). The Newton's method interval single-step procedure for bounding polynomial zeros simultaneously. *Far East Journal of Mathematical Sciences*, 97(2):241–252.

Jamaludin, N. A. A., Monsi, M., Hassan, N., and Masseran, N. (2015b). On convergence of INAAH1 for bounding of polynomial zeros simultaneously. *Far East Journal of Mathematical Sciences*, 98(7):873–881.

Jamaludin, N. A. A., Monsi, M., Hassan, N., and Masseran, N. (2015c). On convergence of INAAH2 procedure for bounding of polynomial zeros simultaneously. *Far East Journal of Mathematical Sciences*, 98(2):259–266.

Monsi, M. (2012). The point symmetric single-step PSS1 procedure for simultaneous approximation of polynomial zeros. *Malaysian Journal of Mathematical Sciences*, 6(1):29–46.

Monsi, M. and Hassan, N. (2015). The repeated procedure PMSS1 for estimating polynomial zeros. *Far East Journal of Mathematical Sciences*, 97(5):645–653.

Monsi, M., Hassan, N., and Rusli, S. F. M. (2012). The point zoro symmetric single-step procedure for simultaneous estimation of polynomial zeros. *Journal of Applied Mathematics*, Article ID 709832.

Monsi, M., Hassan, N., and Rusli, S. F. M. (2014). On modified symmetric single-step procedure for estimating polynomial zeros simultaneously. *International Journal of Mathematical Analysis*, 8(1):27–33.

Monsi, M., Hassan, N., and Rusli, S. F. M. (2015a). The midpoint zoro symmetric single-step procedure for polynomial zeros. *Far East Journal of Mathematical Sciences*, 97(2):231–240.

Monsi, M., Rusli, S. F. M., Hassan, N., Ismail, F., and Ibrahim, Z. B. (2015b). On the convergence of the point repeated symmetric single-step procedure for simultaneous estimation of polynomial zeros. *Malaysian Journal of Mathematical Sciences*, 9(2):301–315.

Monsi, M. and Wolfe, M. A. (1988). An algorithm for the simultaneous inclusion of real polynomial zeros. *Applied Mathematics and Computation*, 25(4):333–346.

Ortega, J. M. and Rheinboldt, W. C. (1970). *Iterative solution of nonlinear equations in several variables.* Academic Press, New York.

Petkovic, M. S. (1982). On an iterative method for simultaneous inclusion of polynomial zeros complex. *Journal of Computational and Applied Mathematics*, 8(1):51–56.

Petkovic, M. S. and Milosevic, D. (2005). Derivative free inclusion methods for polynomial zeros. *Computing*, 75(1):71–83.

Rusli, S. F. M. (2011). Some modifications on interval symmetric single-step procedure for simultaneous inclusion of real zeros of polynomials. Master Thesis at Universiti Putra Malaysia, Malaysia.

Rusli, S. F. M., Monsi, M., Hassan, M. A., and Leong, W. J. (2011). On the interval zoro symmetric single-step procedure for simultaneous finding of real polynomial zeros. *Applied Mathematics Sciences*, 5(75):3693–3706.

Rusli, S. F. M., Monsi, M., and Hassan, N. (2014a). On the convergence of the Newton's method in the interval symmetric single-step procedure. *International Journal of Mathematical Analysis*, 8(56):2775–2779.

Rusli, S. F. M., Monsi, M., and Hassan, N. (2014b). The Newton's method in the interval symmetric single-step procedure for polynomial zeros. *International Journal of Mathematical Analysis*, 8(56):2763–2768.

Rusli, S. F. M., Monsi, M., Hassan, N., and Ali, F. M. (2015a). The repeated procedure IRTSS1 for simultaneous inclusion of polynomial zeros. *Global Journal of Pure and Applied Mathematics*, 11(5):3489–3493.

Rusli, S. F. M., Monsi, M., Hassan, N., and Masseran, N. (2015b). On the convergence of IRTSS1 for simultaneous inclusion of polynomial zeros. *Far East Journal of Mathematical Sciences*, 98(2):251–258.

Rusli, S. F. M., Monsi, M., Hassan, N., Senu, N., Ismail, F., and Ibrahim, Z. B. (2015c). On modified interval repeated zoro symmetric single-step IRZSS1-5D procedure for bounding polynomial zeros simultaneously. *AIP Conf. Proc.*, 1682, 20011.

Sham, A. W. M., Monsi, M., and Hassan, N. (2013a). An efficient interval symmetric single step procedure ISS1-5D for simultaneous bounding of real polynomial zeros. *International Journal of Mathematical Analysis*, 7(20):977–981.

Sham, A. W. M., Monsi, M., and Hassan, N. (2013b). On the interval zoro symmetric single-step procedure IZSS1-5D for the simultaneous bounding of real polynomial zeros. *International Journal of Mathematical Analysis*, 7(59):2947–2951.

Sham, A. W. M., Monsi, M., and Hassan, N. (2015). The interval zoro symmetric single step IZSS1-5D for the simultaneous bounding of real polynomial zeros. *Malaysian Journal of Mathematical Sciences*, 9(2):325–336.

Sham, A. W. M., Monsi, M., Hassan, N., and Suleiman, M. (2013c). A modified interval symmetric single step procedure ISS-5D. *AIP Conf. Proc.*, 1522:61–67.